

# Equilibria in a competitive model arising from linear production situations with a common-pool resource

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## Abstract

In this paper we deal with linear production situations in which there is a limited common-pool resource, managed by an external agent. The profit that a producer, or a group of producers, can attain depends on the amount of common-pool resource obtained through a certain procedure. We contemplate a competitive process among the producers or groups of producers and study the corresponding non cooperative games, describing their (strict) Nash equilibria in pure strategies. It is shown that strict Nash equilibria form a subset of strong Nash equilibria, which in turn form a proper subset of Nash equilibria

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## 1 Introduction

Linear production ( $LP$ ) situations and corresponding cooperative games were introduced in Owen (1975). These are situations where a set of producers own resource bundles that they can use to produce several products through linear production techniques. Their goal is to maximize the profit, which equals the revenue of their products at the given market prices. Tijs et al. (2001) study more general  $LP$  situations involving a countably infinite number of products.

In this paper we deal with linear production situations in which there is a limited common-pool resource ( $LPP$ ), controlled by an external agent and it

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is necessary to produce any product. The motivation to study such a situation is due to the fact that it arises in many real-life situations related to natural resource management such as when the producers need to buy carbon dioxide, water or fish quotas or even to obtain public capital to invest in their firms. This type of situation was recently introduced by Gutiérrez et al. (2015). The profit that a producer, or a group of producers, can attain depends on the amount of common-pool resource obtained through a certain procedure. In Gutiérrez et al. (2015) they consider a collaborative approach to this process. In this paper we contemplate a competitive process among the producers or groups of producers. Depending on the procedure used to obtain the amount of common-pool resource that they need, different games can be defined. In the case of a cooperative process, Gutiérrez et al. (2015) show that these games are partition function form games. They prove that if the common-pool resource is not a constraint for the production process, or it is only a restriction for the grand coalition, then the games reduce to characteristic function games and the games obtained in the former case have a non empty core. In the more general case, when the common-pool resource could be a constraint for the production process, the problem can be modeled as a bankruptcy-like problem. Then, the manager of the common-pool resource can apply bankruptcy rules in order to share this. Bankruptcy techniques have been widely used to deal with scarce resources in many economic problems such as  $k$ -hop minimum cost spanning tree problems (Bergantiños et al, 2012).

In the present work, we consider a competitive process as a mechanism for the producers, or groups of producers, to obtain their share of the common-pool resource. This can be modeled as a non cooperative game among coalitions and we study the existence of Nash equilibria, and some of its refinements, in pure strategies for such a game.

The paper is organized as follows. In Section 2 some concepts on linear production situations with a common-pool resource are presented. In Section 3 we assume that the producers will take part in a competitive process to distribute the common-pool resource. Thus, we study the corresponding non cooperative games, describing their (strict) Nash equilibria, Nash (1950), in pure strategies. Moreover, all strict Nash equilibria turn out to be strong Nash equilibria, Aumann (1959). Section 4 concludes.

## 2 On linear production situations with a common-pool resource

Let  $N = \{1, \dots, n\}$  be a set of producers that have to deal with a linear production problem to produce a set  $G = \{1, \dots, g\}$  of goods from a set  $Q = \{1, \dots, q\}$  of resources. There exists an external agent, called the pool, who has an amount  $r$  of a certain resource that agents need to buy for producing the goods. The model is described by:

- $b^i \in \mathbb{R}_+^q$  are the available resources for each producer  $i \in N$ ,  $b^S = \sum_{i \in S} b^i$ .

$B \in \mathcal{M}_{q \times n}$  is the resource matrix. We assume that there is a positive quantity of each resource, i.e. for all resources  $t \in Q$  there is a producer  $i$  such that  $b_i^t > 0$ .

- $R$  represents the common-pool resource, owned by the pool, whose cost per unit,  $c_R$ , is fixed (exogenously determined) and the total available is denoted by  $r$ .
- $A \in \mathcal{M}_{(q+1) \times g}$  is the production matrix,  $a_{tj}$  is the amount of the resource  $t$  needed to produce the product  $j$ , where the last row is related to the pool-resource and  $a_{(q+1)j} > 0 \quad \forall j \in G$ . Furthermore, we do not allow for output without input and, therefore, for each product  $j \in G$  there is at least one resource  $t \in Q$  with  $a_{tj} > 0$ .
- $p \in \mathbb{R}_{++}^g$  is the price vector. Moreover, in order to face a profitable process we assume that  $p_j > a_{(q+1)j} c_R \quad \forall j \in G$ .

Thus, a linear production situation with a common-pool resource (*LPP*) can be denoted by  $(A, B, p, r, c_R)$ .

The producers can join in groups (coalitions) for the production process, because they can use the same set of productions techniques, and for buying the common-pool resource.

$\mathcal{P}(N)$  represents the set of all partitions of  $N$  and  $P = \{S_1, \dots, S_k\}$  denotes one of these partitions.  $\mathcal{P}_S(N)$  is the set of all partitions of  $N$  that contain  $S$  and  $P_S$  is an element of  $\mathcal{P}_S(N)$ . The profit that a coalition  $S \subseteq N$  can obtain depends on the coalitions formed by the other players  $P_S \in \mathcal{P}_S(N)$  and on what those coalitions do.

We denote by  $value(S; z)$  the value of the linear program:

$$\begin{aligned} \max \quad & \sum_{j=1}^g p_j x_j - c_R z \\ \text{s.t.:} \quad & Ax \leq \begin{pmatrix} b^S \\ z \end{pmatrix} \\ & x \geq \mathbf{0}_g, z \geq 0, \end{aligned} \tag{1}$$

where  $b^S = \sum_{i \in S} b^i$ . The optimal demand of the common-pool resource for each coalition  $S$ ,  $d_S = \min \{z \in \mathbb{R}_+ \mid value(S; z) \text{ is maximum}\}$ , is obtained by solving the linear program (1). Given a partition  $P = \{S_1, \dots, S_k\}$ , its total demand is  $d(P) = \sum_{i=1}^k d_{S_i}$ . One might think that the optimal demands are superadditive, i. e.  $d_S \geq \sum_{i \in S} d_{\{i\}}$ , however, Gutiérrez et al. (2015) show that this is not true. We assume that for all  $S$ , there is a feasible production plan  $(x; z)$  such that  $value(S; z) > 0$ , which implies  $d_S > 0$ .

### 3 A non cooperative game among coalitions

In this section we study the competitive procedure in which the producers take part in order to obtain their share of the common-pool resource.

Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$  and  $d_{S_i}$  the demand of the common-pool resource for each group of producers  $S_i \in P$ . The coalitions in  $P$  proceed to buy, simultaneously, the common-pool resource that they need. A problem arises when the common-pool resource is scarce. In such a situation the coalitions in  $P$  act non cooperatively and try to maximize the amount of common-pool resource. This can be modeled as a non cooperative game among coalitions. To this end, a mechanism of a competitive game where players can be alone or in groups (coalitions) is designed; this takes into account that if they ask for more, in total, than the available amount of the common-pool resource, then the manager (pool) gives them nothing. This can be read as a kind of penalty that the owner of the common-pool resource (carbon dioxide, water or fish quotas) imposes on the producers in order to reach an agreement on a sustainable exploitation of the resource. In this way he forces the self-regulation of the producers. It can be done through other techniques such as dealing with bankruptcy problems as in Gutiérrez et al. (2015).

More formally, assume that each coalition  $S_i \in P$  chooses an amount  $z_i$  of the common-pool resource to buy, i.e. its set of strategies is  $X_i = [0, d_{S_i}]$  and its profit is given by

$$\pi_i(z_1, \dots, z_k) = \begin{cases} 0, & \text{if } \sum_{i=1}^k z_i > r \\ \text{value}(S_i; z_i), & \text{otherwise.} \end{cases}$$

We denote by  $(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  the non cooperative game played by coalitions in  $P$  when they try to obtain the highest amount of the common-pool resource. The concept of equilibrium as defined by Nash (1950) and its refinements are the most widely used solution concepts in non cooperative games.

**Definition 1** Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ ,  $z^* \in \prod_{i=1}^k X_i$  is a Nash equilibrium (NE) in pure strategies of the non cooperative game  $(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  if for all  $i \in \{1, \dots, k\}$

$$\pi_i(z_1^*, \dots, z_{i-1}^*, z_i^*, z_{i+1}^*, \dots, z_k^*) \geq \pi_i(z_1^*, \dots, z_{i-1}^*, z_i, z_{i+1}^*, \dots, z_k^*),$$

for all  $z_i \in X_i$ .

**Definition 2** Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ ,  $z^* \in \prod_{i=1}^k X_i$  is a strict Nash equilibrium (sNE) in pure strategies of the non cooperative game  $(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  if for all  $i \in \{1, \dots, k\}$

$$\pi_i(z_1^*, \dots, z_{i-1}^*, z_i^*, z_{i+1}^*, \dots, z_k^*) > \pi_i(z_1^*, \dots, z_{i-1}^*, z_i, z_{i+1}^*, \dots, z_k^*),$$

for all  $z_i \in X_i \setminus \{z_i^*\}$ .

Aumann (1959) proposed the idea of strong Nash equilibrium (SNE), defined as a strategic profile for which no subset of players has a joint deviation that strictly benefits all of them, while all other players maintain their equilibrium strategies.

**Definition 3** Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ ,  $z^* \in \prod_{i=1}^k X_i$  is a strong Nash equilibrium (SNE) in pure strategies of the non cooperative game  $(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  if for all  $M \subset P$  there does not exist any  $z_M \in X_M$  such that

$$\pi_i(z_M, z_{-M}^*) > \pi_i(z^*), \forall i \in M,$$

where  $X_M = \prod_{i \in M} X_i$ .

Firstly, we study the less controversial case, when the common-pool resource is sufficient to meet demands.

**Theorem 4** Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ ,  $(d_{S_1}, \dots, d_{S_k})$  is the only Nash equilibrium of the non cooperative game  $(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  if and only if  $d(P) \leq r$ .

**Proof.** First, we prove the sufficient part. Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$  such that  $d(P) \leq r$ . Then, by the definition of  $d_{S_i}$ , it holds  $\pi_i(d_{S_1}, \dots, d_{S_k}) = \text{value}(S_i; d_{S_i}) \geq \text{value}(S_i; z) = \pi_i(d_{S_1}, \dots, d_{S_{i-1}}, z_i, d_{S_{i+1}}, \dots, d_{S_k})$  for all  $z_i \leq d_{S_i}$ , and so  $(d_{S_1}, \dots, d_{S_k})$  is a Nash equilibrium.

Let us assume that there is another Nash equilibrium  $(z_1, \dots, z_k)$ . Thus, there is at least one  $i$  such that  $z_i < d_{S_i}$ , but from the definition we know that  $d_{S_i} = \min\{z \in R_+ : \text{value}(S_i; z) \text{ is maximum}\}$ . Therefore, we have that  $\text{value}(S_i; z_i) < \text{value}(S_i; d_{S_i})$ , which contradicts the fact that  $(z_1, \dots, z_k)$  is a Nash equilibrium.

Next, we prove the necessary part. Let  $(d_{S_1}, \dots, d_{S_k})$  be the unique Nash equilibrium. Let us assume that  $d(P) > r$ . This implies  $\pi_i(d_{S_1}, \dots, d_{S_k}) = 0, \forall i \in \{1, \dots, k\}$ . We will distinguish three possible situations:

1.  $\sum_{i \neq j} d_{S_i} > r, \forall j \in \{1, \dots, k\}$ .

Let  $\varepsilon > 0$  be such that  $\varepsilon \leq \min_{j \in \{1, \dots, k\}} \left\{ \sum_{i \neq j} d_{S_i} \right\}$  and  $(k-1)\varepsilon < \min_{j \in \{1, \dots, k\}} \left\{ \sum_{i \neq j} d_{S_i} - r \right\}$ . Then,  $(d_{S_1} - \varepsilon, \dots, d_{S_k} - \varepsilon)$  is also a Nash equilibrium. Namely,  $\pi_i(d_{S_1} - \varepsilon, \dots, d_{S_k} - \varepsilon) = 0, \forall i \in \{1, \dots, k\}$ . On the other hand,  $\sum_{i \neq j} (d_{S_i} - \varepsilon) = \sum_{i \neq j} d_{S_i} - (k-1)\varepsilon > r, \forall j \in \{1, \dots, k\}$ . Therefore, for all  $i \in \{1, \dots, k\}$ , it holds

$$0 = \pi_i(d_{S_1} - \varepsilon, \dots, d_{S_j} - \varepsilon, \dots, d_{S_k} - \varepsilon) \geq \pi_i(d_{S_1} - \varepsilon, \dots, z_i, \dots, d_{S_k} - \varepsilon) = 0,$$

$\forall z_i \in [0, d_{S_i}]$ . Then, this contradicts the uniqueness of the Nash equilibrium.

2.  $\exists j \in \{1, \dots, k\}$  such that  $\sum_{i \neq j} d_{S_i} < r$ .

We can choose  $z_j \leq r - \sum_{i \neq j} d_{S_i}$  such that  $\text{value}(S_j; z_j) > 0$ , since we know by hypothesis that there are positive profits. Now, for  $(d_{S_1}, \dots, z_j, \dots, d_{S_k})$  we have that

$$\text{value}(S_j; z_j) = \pi_j(d_{S_1}, \dots, z_j, \dots, d_{S_k}) > \pi_j(d_{S_1}, \dots, d_{S_i}, \dots, d_{S_k}),$$

which is a contradiction because  $(d_{S_1}, \dots, d_{S_k})$  is a Nash equilibrium by hypothesis.

3.  $\nexists j \in \{1, \dots, k\}$  such that  $\sum_{i \neq j} d_{S_i} < r$ .

All the above implies that  $\sum_{i \neq j} d_{S_i} \geq r, \forall j \in \{1, \dots, k\}$  with at least one equality in these expressions, because otherwise we would be in case 1 and this is impossible.

Let us assume that for  $j$  we have  $\sum_{i \neq j} d_{S_i} = r$ . Then,  $\left(d_{S_1}, \dots, \underbrace{0}_j, \dots, d_{S_k}\right)$  is a Nash equilibrium since

$$0 = \pi_j \left(d_{S_1}, \dots, \underbrace{0}_j, \dots, d_{S_k}\right) \geq \pi_j(d_{S_1}, \dots, z_j, \dots, d_{S_k}) = 0, \forall z_j \in [0, d_{S_j}],$$

since  $\sum_{i \neq j} d_{S_i} + z_j > r, \forall z_j > 0$ .

Likewise, for all  $i \neq j$  we know that,  $\forall z_i \in [0, d_{S_i}]$ ,

$$\begin{aligned} \pi_i \left(d_{S_1}, \dots, \underbrace{0}_j, \dots, d_{S_k}\right) &= \text{value}(S_i; d_{S_i}) \geq \\ \text{value}(S_i; z_i) &= \pi_i \left(d_{S_1}, \dots, z_i, \dots, \underbrace{0}_j, \dots, d_{S_k}\right), \end{aligned}$$

by definition of  $d_{S_i}$  and  $\sum_{i \neq j} d_{S_i} = r$ . But, this is a contradiction with the uniqueness of  $(d_{S_1}, \dots, d_{S_k})$  as a Nash equilibrium. ■

The following result shows an analysis, in terms of Nash equilibria, when the amount of the common-pool resource  $R$  is not enough to meet the demand expectations of the players, which are grouped in different coalitions that form a partition of the whole set. This covers all the possibilities that may appear in these situations.

**Theorem 5** *Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ . If  $d(P) > r$ , then the set of all Nash equilibria is given by*

$$NE(X_1, \dots, X_k; \pi_1, \dots, \pi_k) = \left\{ z \in \prod_{i=1}^k [0, d_{S_i}] : \begin{array}{l} \sum_{i=1}^k z_i = r, \text{ or} \\ \sum_{i \neq j} z_i > r, \forall j \in \{1, \dots, k\}, \text{ or} \\ \sum_{i \neq j} z_i = r, \forall j \in \{1, \dots, k\}, z_i = \frac{r}{k-1}, z_i \leq d_{S_i} \end{array} \right\}.$$

**Proof.** It easy to check that they are Nash equilibria. In the first case, we should take into account that for  $z \in \prod_{i=1}^k [0, d_{S_i}]$  with  $\sum_{i=1}^k z_i = r$  if  $z_j, z'_j \in [0, d_{S_j}]$  are such that  $z'_j < z_j$ , then  $value(S_j; z'_j) \leq value(S_j; z_j)$ . This result holds because  $value(S_j; z'_j) < value(S_j; d_{S_j})$ , due to the uniqueness of  $d_{S_j}$ ,  $z'_j < d_{S_j}$  and  $z_j = \alpha z'_j + (1 - \alpha)d_{S_j}$  with  $\alpha > 0$ . Thus,  $value(S_j; z_j) \geq \alpha value(S_j; z'_j) + (1 - \alpha)value(S_j; d_{S_j}) > value(S_j; z'_j)$ . The remaining cases are obvious because unilateral deviations do not produce any benefit. Let us see that they are the complete list of all Nash equilibria. To this end, we will distinguish several cases.

1. Let us assume that there is a Nash equilibrium  $z \in \prod_{i=1}^k [0, d_{S_i}]$  such that  $\sum_{i=1}^k z_i < r$ . Then, there exists  $j \in \{1, \dots, k\}$  such that  $z_j < d_{S_j}$ , otherwise  $z_i = d_{S_i}, \forall i \in \{1, \dots, k\}$ , and  $\sum_{i=1}^k z_i = \sum_{i=1}^k d_{S_i} > r$ . Nevertheless, in the former case,  $j$  has incentives to deviate and choose  $z'_j > z_j$  with  $\sum_{i \neq j}^k z_i + z'_j \leq r$  and  $z'_j \leq d_{S_j}$ , so it would not be a Nash equilibrium.

2. Let us assume that there is a Nash equilibrium  $z \in \prod_{i=1}^k [0, d_{S_i}]$  such that  $\sum_{i=1}^k z_i > r$ . Let us note that, in this case  $\pi_i(z) = 0$  for all  $i \in \{1, \dots, k\}$ .

- 2.1 If there is a  $j \in \{1, \dots, k\}$  such that  $\sum_{i \neq j} z_i < r$ , then if  $j$  deviates from

$$z_j \text{ to } r - \sum_{i \neq j} z_i \text{ she will obtain an amount } \pi_j(z_1, \dots, r - \sum_{i \neq j} z_i, \dots, z_k) = value\left(S_j; r - \sum_{i \neq j} z_i\right) > 0. \text{ Therefore, } z \text{ is not a Nash equilibrium.}$$

- 2.2 If  $\sum_{i \neq j} z_i = r, \forall j \in \{1, \dots, k\}$ , then no  $j$  has incentive to deviate. Never-

theless, for all  $j \in \{1, \dots, k\}$ ,  $\sum_{i=1}^k z_i = \sum_{i \neq j} z_i + z_j = r + z_j$ . If we add in

$$\begin{aligned} k \sum_{i=1}^k z_i &= kr + \sum_{j=1}^k z_j \Rightarrow \sum_{i=1}^k z_i = r + \frac{1}{k} \sum_{i=1}^k z_i \Rightarrow \\ z_j &= \frac{1}{k} \sum_{i=1}^k z_i = K, \forall j \in \{1, \dots, k\} \text{ and } K \in \mathbb{R} \Rightarrow \\ r &= \sum_{i \neq j} z_i = (k-1)K \Rightarrow K = \frac{r}{k-1}. \end{aligned}$$

- 2.2.1 If there is  $i$  such that  $\frac{r}{k-1} > d_{S_i}$ , then we have a contradiction with  $z_i \in [0, d_{S_i}]$ .

- 2.2.2 If for all  $i$  we have that  $\frac{r}{k-1} \leq d_{S_i}$ , then  $z$  is a Nash equilibrium.

■

The next result illustrates which is the set of strict Nash equilibria, when the amount of the common-pool resource  $R$  is not enough to meet the demand expectations of the players.

**Theorem 6** *Let  $P = \{S_1, \dots, S_k\}$  be a partition of  $N$ . If  $d(P) > r$ , then the set of all strict Nash equilibria is given by*

$$sNE(X_1, \dots, X_k; \pi_1, \dots, \pi_k) = \left\{ z \in \prod_{i=1}^k [0, d_{S_i}] : \sum_{i=1}^k z_i = r \text{ and } \forall i \in \{1, \dots, k\}, z_i > 0 \right\}. \quad (2)$$

**Proof.**

1. Every strategy vector in  $sNE(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  is obviously a Nash equilibrium by Theorem 5, since

$$sNE(X_1, \dots, X_k; \pi_1, \dots, \pi_k) \subseteq NE(X_1, \dots, X_k; \pi_1, \dots, \pi_k).$$

2. Every strategy vector in  $sNE(X_1, \dots, X_k; \pi_1, \dots, \pi_k)$  is a strict Nash equilibrium. We will distinguish two cases:

- 2.1  $\forall z'_i < z_i$  we have that  $value(S_i; z'_i) < value(S_i; z_i)$ . The result holds because  $value(S_i; z'_i) < value(S_i; d_{S_i})$  due to the uniqueness of  $d_{S_i}$ ,  $z'_i < d_{S_i}$  and  $z_i = \alpha z'_i + (1 - \alpha)d_{S_i}$  with  $\alpha > 0$ . Thus,  $value(S_i; z_i) \geq \alpha value(S_i; z'_i) + (1 - \alpha)value(S_i; d_{S_i}) > value(S_i; z'_i)$ .
- 2.2  $\forall z'_i > z_i$  we know that  $\sum_{j \neq i} z_j + z'_i > r$ , which implies  $\pi_{S_i} = 0$  by definition and  $value(S_i; z_i) > 0$ .

3. Let us assume that there is at least one strict Nash equilibria different to those in (2). Then, from Theorem 5, we should consider several cases:

- 3.1  $\sum_{i=1}^k z_i > r$ . By Theorem 5 we know that  $z$  is a Nash equilibrium. But, obviously, it is not strict because  $\pi_i = 0$  by definition of the payoffs.
- 3.2  $\sum_{i=1}^k z_i = r$  and there is at least a coalition  $S_j$  such that  $z_j = 0$ . On the one hand, we know from Theorem 5 that a Nash equilibrium exists, on the other hand, we have that  $\pi_j(z_j) = 0$  and  $\pi_j(z'_j) = 0$ ,  $\forall z'_j > z_j$ . Therefore, this is a Nash equilibrium which is not strict. ■

Additionally, all strict Nash equilibria in (2) are strong Nash equilibria. Because following a similar reasoning as that used in the previous result, it is not possible that several producers deviate together in such a way that all of them are strictly better off. Furthermore, it is easy to check that  $z \in \prod_{i=1}^k [0, d_{S_i}]$



such that  $\sum_{i=1}^k z_i = r$  with  $z_j = 0$ , for some  $j \in \{1, \dots, k\}$ , is a strong Nash equilibrium which is not strict. Therefore, strict Nash equilibria form a proper subset of strong Nash equilibria, which in turn form a proper subset of Nash equilibria. Thus, we do not characterize the strong Nash equilibria because we have described the strict Nash equilibria which are in a more restrictive class.

## 4 Concluding remarks

In view of these results we conclude that the set of possible outcomes is very large. So, to decide whether we will find one or another is difficult. What seems clear is that the fairest Nash equilibria and most advantageous for the market are those that distribute the whole resource, when this is scarce, among all participants.

We should point out that the coordination among coalitions is important in order not to receive a zero amount of the common-pool resource. Another possibility to deal with the situation where the players ask for more than the available amount of the common-pool resource, is to use bankruptcy techniques as in Gutiérrez et al. (2015).

We have assumed throughout the paper that every producer knows the resources of the others. We leave for further research the study, in terms of Bayesian equilibria, when this is not the case. Moreover, we have considered that the price of the common-pool resource is fixed, but we could explore the case when it depends on its own demand. Furthermore, we would like to study the auction problem that arises when the total amount of the common-pool resource is auctioned among the coalitions in a partition, where each coalition bids an amount to buy and a price to pay per unit of the common-pool resource as the model of the electricity market described in Sancho et al. (2008).

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